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Electromagnetic field generated by a charge moving along a helical orbit inside a dielectric cylinder

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Abstract
The electromagnetic field generated by a charged particle moving along a helical orbit inside a dielectric cylinder immersed into a homogeneous medium is investigated. Expressions are derived for the electromagnetic potentials, electric and magnetic fields in the region inside the cylinder. The parts corresponding to the radiation field are separated. The radiation intensity on the lowest azimuthal mode is studied.

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1. Introduction

The radiation from a charged particle moving along a helical orbit in vacuum has been widely discussed in literature (see, for instance, [1, 2] and references given therein). This type of electron motion is used in helical undulators for generating electromagnetic radiation in a narrow spectral interval at frequencies ranging from radio or millimetre waves to x-rays. The unique characteristics, such as high intensity and high collimation, have resulted in extensive applications of this radiation in a wide variety of experiments and in many disciplines. These applications motivate the importance of investigations for various mechanisms of controlling the radiation parameters. From this point of view, it is of interest to consider the influence of a medium on the spectral and angular distributions of the radiation.

It is well known that the presence of medium can essentially change the characteristics of the high-energy electromagnetic processes and gives rise to new types of phenomena. Well-known examples are Cherenkov, transition, and diffraction radiations. In a series of papers [3–11] the synchrotron radiation is considered from a charge rotating around a dielectric ball/cylinder enclosed by a homogeneous medium. It has been shown that the superposition between the synchrotron and Cherenkov radiations leads to interesting effects: under the Cherenkov condition for the material of the ball/cylinder and the particle velocity, strong narrow peaks appear in the radiation intensity. At these peaks the radiated energy exceeds the corresponding quantity in the case of a homogeneous medium by several orders of magnitude. The influence of a dielectric cylinder on the radiation from a longitudinal charged oscillator moving with constant drift velocity along the axis of the cylinder has been considered in [12, 13].

In a previous paper [14] we have investigated the radiation by a charged particle moving along a helical orbit inside a dielectric cylinder immersed into a homogeneous medium (for the radiation in a dispersive homogeneous medium see [15]). Specifically, formulae are derived for the electromagnetic fields and for the spectral-angular distribution of the radiation intensity in the exterior medium. A special case of the relativistic motion along the direction of the cylinder axis with non-relativistic transverse velocity is discussed in detail and various regimes for the undulator parameter are considered. In the present paper we consider the electromagnetic fields generated inside the cylinder by the charge moving along a helical orbit. The paper is organized as follows. In the next section, by using the Green function, the vector potential and electromagnetic fields are determined for the region inside the cylinder. The analytic structure of the Fourier components of the fields are investigated in section 3 and the radiation part of the field is separated. A formula is derived for the radiation intensity on the lowest azimuthal mode. Section 4 concludes the main results of the paper. In the appendix we derive a summation formula over the eigenmodes of the dielectric cylinder which is used in section 3 for the evaluation of the mode sums in the formulae of the radiation intensities.

2. Electromagnetic potentials and fields inside a cylinder

We consider a dielectric cylinder of radius ρ_1 and with dielectric permittivity ε_0 and a point charge q moving along the helical trajectory of radius $\rho_0 < \rho_1$. We assume that the system is immersed in a homogeneous medium with permittivity ε_1. The velocities of the charge along the axis of the cylinder (drift velocity) and in the perpendicular plane we will denote by $v_\parallel$ and $v_\perp$, respectively. In a properly chosen cylindrical coordinate system (ρ, ϕ, z) with the z-axis along the cylinder axis, the components of the current density created by the charge are given by the formula

$$j_l = \frac{q}{\rho} v_l \delta(\rho - \rho_0)\delta(\phi - \omega_0 t)\delta(z - v_\parallel t), \tag{1}$$

where $\omega_0 = v_\perp/\rho_0$ is the angular velocity of the charge. This type of motion can be produced by a uniform constant magnetic field directed along the axis of a cylinder, by a circularly polarized plane wave, or by a spatially periodic transverse magnetic field of constant absolute value and a direction that rotates as a function of the coordinate z. In the helical undulators the last configuration is used with the magnetic field given by (in Cartesian coordinates): $\mathbf{H}_\mathrm{u} = H_\mathrm{u}(-\sin(k_\mathrm{u} z), \cos(k_\mathrm{u} z), 0)$, where $k_\mathrm{u} = 2\pi/\lambda_\mathrm{u}$ and λ_u is the undulator period length. The corresponding parameters for the particle orbit are related to the particle velocity v and to the undulator parameters by the formulae

$$v_\parallel = \sqrt{v^2 - \frac{q^2 H_\mathrm{u}^2}{\mathcal{E}^2 k_\mathrm{u}^2}}, \qquad \omega_0 = k_\mathrm{u} v_\parallel, \qquad \rho_0 = \frac{q H_\mathrm{u}}{\mathcal{E} k_\mathrm{u} \omega_0}, \tag{2}$$

with $\mathcal{E}$ being the particle energy.

The vector potential of the electromagnetic field is expressed in terms of the Green function $G_{il}(\mathbf{r}, t, \mathbf{r}', t')$ as

$$A_i(\mathbf{r}, t) = -\frac{1}{2\pi^2 c} \int G_{il}(\mathbf{r}, t, \mathbf{r}', t') j_l(\mathbf{r}', t)\, \mathrm{d}\mathbf{r}'\, \mathrm{d}t', \tag{3}$$

where the summation over l is understood. For static and cylindrically symmetric medium the Green function is presented in the form of the Fourier expansion

$$G_{il}(\mathbf{r},t,\mathbf{r}',t') = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}k_z\,\mathrm{d}\omega G_{il}(m,k_z,\omega,\rho,\rho') \times \exp[\mathrm{i}m(\phi-\phi')+\mathrm{i}k_z(z-z')-\mathrm{i}\omega(t-t')]. \tag{4}$$

The substitution of expressions (1) and (4) into formula (3) gives the following result for the components of the vector potential:

$$A_l(\mathbf{r},t) = -\frac{q}{\pi c}\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\omega_0 t)}\int_{-\infty}^{\infty}\mathrm{d}k_z\,\mathrm{e}^{\mathrm{i}k_z(z-v_\parallel t)} \times [v_\perp G_{l\phi}(m,k_z,\omega_m(k_z),\rho,\rho_0)+v_\parallel G_{lz}(m,k_z,\omega_m(k_z),\rho,\rho_0)], \tag{5}$$

where

$$\omega_m(k_z) = m\omega_0 + k_z v_\parallel. \tag{6}$$

In the Lorentz gauge, using the formula for the Green function given earlier in [3] and introducing the notations

$$\lambda_j^2 = \frac{\omega_m^2(k_z)}{c^2}\varepsilon_j - k_z^2, \qquad j=0,1, \tag{7}$$

for the corresponding Fourier components $G_{il} = G_{il}(m,k_z,\omega_m(k_z),\rho,\rho_0)$ in the region inside the cylinder, $\rho < \rho_1$, we obtain

$$G_{l\phi} = \frac{\pi}{4\mathrm{i}^{\sigma_l-1}}\sum_{p=\pm1}p^{\sigma_l}\left[J_{m+p}(\lambda_0\rho_<)H_{m+p}(\lambda_0\rho_>)+B_m^{(p)}J_{m+p}(\lambda_0\rho)\right], \qquad l=\rho,\phi, \tag{8a}$$

$$G_{lz} = -\frac{k_z}{2\mathrm{i}^{\sigma_l}}\frac{J_m(\lambda_0\rho_0)H_m(\lambda_1\rho_1)}{\rho_1\alpha_m W_m^J}\sum_{p=\pm1}\frac{J_{m+p}(\lambda_0\rho)}{p^{\sigma_l-1}W_{m+p}^J}H_{m+p}(\lambda_1\rho_1), \qquad l=\rho,\phi, \tag{8b}$$

$$G_{zz} = \frac{\pi}{2\mathrm{i}}\left[J_m(\lambda_0\rho_<)H_m(\lambda_0\rho_>) - J_m(\lambda_0\rho_0)J_m(\lambda_0\rho)\frac{W_m^H}{W_m^J}\right], \tag{8c}$$

where $J_m(x)$ is the Bessel function and $H_m(x) = H_m^{(1)}(x)$ is the Hankel function of the first kind, $\rho_< = \min(\rho,\rho_0)$, $\rho_> = \max(\rho,\rho_0)$, and $\sigma_\rho = 1$, $\sigma_\phi = 2$. The coefficients $B_m^{(p)}$ in these formulae are determined by the expressions

$$B_m^{(p)} = -J_{m+p}(\lambda_0\rho_0)\frac{W_{m+p}^H}{W_{m+p}^J} + \frac{\mathrm{i}p\lambda_1 H_{m+p}(\lambda_1\rho_1)}{\pi\rho_1\alpha_m W_{m+p}^J}H_m(\lambda_1\rho_1)\sum_{l=\pm1}\frac{J_{m+l}(\lambda_0\rho_0)}{W_{m+l}^J}, \tag{9}$$

where

$$\alpha_m = \frac{\varepsilon_0}{\varepsilon_1-\varepsilon_0} - \frac{1}{2}\lambda_0 J_m(\lambda_0\rho_1)\sum_{l=\pm1}l\frac{H_{m+l}(\lambda_1\rho_1)}{W_{m+l}^J}. \tag{10}$$

Here and below we use the notation

$$W_m^F = F_m(\lambda_0\rho_1)\frac{\partial H_m(\lambda_1\rho_1)}{\partial\rho_1} - H_m(\lambda_1\rho_1)\frac{\partial F_m(\lambda_0\rho_1)}{\partial\rho_1}. \tag{11}$$

with $F = J, H$. Taking $\varepsilon_1 = \varepsilon_0$, from (8) we obtain the Fourier components of the Green function in a homogeneous medium. In this limit $B_m^{(p)} = 0$ and $W_m^H = 0$. By using formulae (8), the electromagnetic fields can be evaluated for an arbitrary motion of a charged particle with zero radial velocity.

In the case of the motion along helical orbit, substituting the expressions for the components of the Green function into formula (5), we present the vector potential in the form of the Fourier expansion

$$A_l(\mathbf{r},t)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\omega_0 t)}\int_{-\infty}^{\infty}\mathrm{d}k_z\,\mathrm{e}^{\mathrm{i}k_z(z-v_\parallel t)}A_{ml}(k_z,\rho). \tag{12}$$

As the functions $A_l(\mathbf{r},t)$ are real one has $A^*_{ml}(k_z,\rho)=A_{-ml}(-k_z,\rho)$. Consequently, formula (12) can also be rewritten in the form

$$A_l(\mathbf{r},t)=2\,\mathrm{Re}\sum_{m=0}^{\infty}{}'\,\mathrm{e}^{\mathrm{i}m(\phi-\omega_0 t)}\int_{-\infty}^{\infty}\mathrm{d}k_z\,\mathrm{e}^{\mathrm{i}k_z(z-v_\parallel t)}A_{ml}(k_z,\rho), \tag{13}$$

where the prime means that the term $m=0$ should be taken with the weight $1/2$. In the discussion below we will assume that $m\geqslant 0$. The Fourier components $A_{ml}=A_{ml}(k_z,\rho)$ can be presented in the form

$$A_{ml}=A^{(0)}_{ml}+A^{(1)}_{ml}, \tag{14}$$

where $A^{(0)}_{ml}$ corresponds to the field of the charge in a homogeneous medium with permittivity ε_0. For this part one has

$$A^{(0)}_{ml}=-\frac{qv_\perp}{4c\mathrm{i}^{\sigma_l-1}}\sum_{p=\pm1}p^{\sigma_l}J_{m+p}(\lambda_0\rho_<)H_{m+p}(\lambda_0\rho_>),\qquad l=\rho,\phi, \tag{15a}$$

$$A^{(0)}_{mz}=\frac{qv_\parallel\mathrm{i}}{2c}J_m(\lambda_0\rho_<)H_m(\lambda_0\rho_>). \tag{15b}$$

The part due to the presence of the cylinder is determined by the formulae

$$A^{(1)}_{ml}=\frac{q}{2\pi\mathrm{i}^{\sigma_l}}\sum_{p=\pm1}p^{\sigma_l}C^{(p)}_m J_{m+p}(\lambda_0\rho),\qquad l=\rho,\phi, \tag{16a}$$

$$A^{(1)}_{mz}=-\frac{\mathrm{i}qv_\parallel}{2c}\frac{W^H_m}{W^J_m}J_m(\lambda_0\rho_0)J_m(\lambda_0\rho), \tag{16b}$$

with the coefficients

$$C^{(p)}_m=-\frac{\mathrm{i}\pi v_\perp}{2c}B^{(p)}_m+pv_\parallel k_z\frac{J_m(\lambda_0\rho_0)H_{m+p}(\lambda_1\rho_1)H_m(\lambda_1\rho_1)}{c\rho_1\alpha_m W^J_m W^J_{m+p}}. \tag{17}$$

The electric and magnetic fields are obtained by means of standard formulae of electrodynamics. As is seen from formula (12), analogous expressions may also be written for these fields. For the Fourier components of the magnetic field one has

$$H_{ml}(k_z,\rho)=H^{(0)}_{ml}+H^{(1)}_{ml}, \tag{18}$$

where $H^{(0)}_{ml}$ corresponds to the field of the charge in a homogeneous medium with permittivity ε_0. For $\rho>\rho_0$ this part is determined by the formulae

$$H^{(0)}_{ml}=-\frac{qk_z}{2\pi\mathrm{i}^{\sigma_l}}\sum_{p=\pm1}p^{\sigma_l-1}D^{(0p)}_m H_{m+p}(\lambda_0\rho),\qquad l=\rho,\phi, \tag{19a}$$

$$H^{(0)}_{mz}=-\frac{q\lambda_0}{2\pi}\sum_{p=\pm1}pD^{(0p)}_m H_m(\lambda_0\rho), \tag{19b}$$

with the coefficients

$$D_m^{(0p)} = \frac{\pi}{2\mathrm{i}c}\left[v_\perp J_{m+p}(\lambda_0\rho_0) - v_\parallel \frac{\lambda_0}{k_z} J_m(\lambda_0\rho_0)\right]. \tag{20}$$

The corresponding expressions for $\rho < \rho_0$ are obtained from (19) by the replacements $J \rightleftarrows H$ of the Bessel and Hankel functions. The part $H_{ml}^{(1)}$ in (18) is due to the inhomogeneity and is given by formulae

$$H_{ml}^{(1)} = -\frac{qk_z}{2\pi \mathrm{i}^{\sigma_l}} \sum_{p=\pm 1} p^{\sigma_l - 1} D_m^{(p)} J_{m+p}(\lambda_0\rho), \qquad l = \rho, \phi, \tag{21a}$$

$$H_{mz}^{(1)} = -\frac{q\lambda_0}{2\pi} \sum_{p=\pm 1} p D_m^{(p)} J_m(\lambda_0\rho), \tag{21b}$$

where the notation

$$D_m^{(p)} = C_m^{(p)} - \frac{\mathrm{i}\pi v_\parallel \lambda_0}{2ck_z} \frac{W_m^H}{W_m^J} J_m(\lambda_0\rho_0), \qquad p = \pm 1, \tag{22}$$

is introduced. By making use of the Maxwell equation $\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon_0 \mathbf{E}/c$, one can derive the corresponding Fourier coefficients for the electric field:

$$E_{ml}^{(1)} = \frac{qc\mathrm{i}^{1-\sigma_l}}{4\pi\omega_m(k_z)\varepsilon_0} \sum_{p=\pm 1} p^{\sigma_l} J_{m+p}(\lambda_0\rho)\left[\left(\frac{\omega_m^2(k_z)\varepsilon_0}{c^2} + k_z^2\right) D_m^{(p)} - \lambda_0^2 D_m^{(-p)}\right], \tag{23a}$$

$$E_{mz}^{(1)} = \frac{q\mathrm{i}c\lambda_0 k_z}{2\pi\omega_m(k_z)\varepsilon_0} \sum_{p=\pm 1} D_m^{(p)} J_m(\lambda_0\rho), \tag{23b}$$

where $l = \rho, \phi$. As it follows from these formulae, $\mathbf{E}_m^{(1)} \cdot \mathbf{H}_m^{(1)} = 0$, i.e., the corresponding Fourier components of the electric and magnetic fields are perpendicular to each other.

Taking in the formulae given above $\omega_0 = 0$ for a fixed ρ_0, as a special case we obtain the fields generated by a charge moving with a constant velocity $v_\parallel$ on a straight line $\rho = \rho_0$ parallel to the cylinder axis. For this case one has

$$\lambda_i^2 = \lambda_i^{(0)2} \equiv k_z^2\left(\beta_{i\parallel}^2 - 1\right), \qquad \beta_{i\parallel} = v_\parallel\sqrt{\varepsilon_i}/c, \qquad i = 0, 1, \tag{24}$$

and $v_\perp = 0$. As a result the dependence on the parameter ρ_0 in the coefficients $D_m^{(p)}$ is in the form of the Bessel function $J_m(\lambda_0\rho_0)$. It follows from here that in the limit $\rho_0 \to 0$ (the particle moves along the axis of the dielectric cylinder) the term with $m = 0$ contributes only. The latter property is a simple consequence of the azimuthal symmetry of the corresponding problem.

The formulae for the fields are simplified for the $m = 0$ mode. In this case $\lambda_i = \lambda_i^{(0)}$ and for the function α_m from (10) one has

$$\alpha_0 = \frac{\varepsilon_1\lambda_0 J_0(\lambda_0\rho_1)H_0'(\lambda_1\rho_1) - \varepsilon_0\lambda_1 J_0'(\lambda_0\rho_1)H_0(\lambda_1\rho_1)}{(\varepsilon_1 - \varepsilon_0)W_1^J}. \tag{25}$$

The expression for the coefficients $D_0^{(p)}$ takes the form

$$D_0^{(p)} = p\frac{\mathrm{i}\pi v_\perp W_1^H}{2cW_1^J} J_1(\lambda_0\rho_0) - \frac{v_\parallel J_0(\lambda_0\rho_0)}{cW_0^J}\left[\frac{\mathrm{i}\pi\lambda_0}{2k_z}W_0^H + \frac{k_z}{\rho_1}\frac{H_0(\lambda_1\rho_1)H_0'(\lambda_1\rho_1)}{\alpha_0 W_1^J}\right]. \tag{26}$$

For the corresponding Fourier components of the electric field we obtain the formulae

$$E_{0l}^{(1)} = \frac{qck_z\beta_{0\parallel}^{2(\sigma_l-1)}}{2\pi\mathrm{i}^{\sigma_l-1}v_\parallel\varepsilon_0} J_1(\lambda_0\rho) \sum_{p=\pm 1} p^{\sigma_l-1} D_0^{(p)}, \qquad E_{0z}^{(1)} = \frac{\mathrm{i}qc\lambda_0^{(0)}k_z}{2\pi v_\parallel\varepsilon_0} J_0(\lambda_0\rho) \sum_{p=\pm 1} D_0^{(p)}, \tag{27}$$

with $l = \rho, \phi$ and σ_l is defined after formulae (8). These formulae are used in the next section to derive the formula for the corresponding radiation intensity inside the cylinder.

3. Radiation fields inside a dielectric cylinder

In this section we consider the radiation field propagating inside a dielectric cylinder. First of all let us show that the part corresponding to the fields of the charge in a homogeneous medium with permittivity ε_0 does not contribute to the radiation field in this region. This directly follows from the estimate of the integral over k_z in the corresponding formula (12) on the base of the stationary phase method. As in the integral over k_z the phase $k_z z$ has no stationary points and the integrand is a function of the class $C^\infty(R)$, for large values $|z|$ the integral vanishes more rapidly than any power of $1/|z|$. From this argument it follows that the radiation field is determined by the singular points of the integrand in the integral over k_z. In the expressions for the parts of the Fourier components $H_{ml}^{(1)}$ and $E_{ml}^{(1)}$, the coefficients $D_m^{(p)}$ enter in the form of combinations $\sum_p D_m^{(p)}$ and $\sum_p pD_m^{(p)}$. By using formulae (22), it can be seen that for $m \neq 0$ these combinations are regular at the points corresponding to the zeros of the functions W_m^J and $W_{m\pm 1}^J$. The only poles of the parts of the Fourier components for the fields determined by relations (21), (23) are the zeros of the function α_m appearing in the denominators of equations (9), (17). It can be seen that this function has zeros only for $\lambda_1^2 < 0$. Note that for the corresponding modes in the exterior region (see [14]), $\rho > \rho_1$, the Fourier coefficients are proportional to the MacDonald function $K_\nu(|\lambda_1|\rho)$, $\nu = m, m \pm 1$, and they are exponentially damped in the region outside the cylinder. These modes are the eigenmodes of the dielectric cylinder and propagate inside the cylinder. By using the properties of the cylindrical functions, for $m \neq 0$ the function α_m can also be written in the form

$$\alpha_m = \frac{U_m}{(\varepsilon_1 - \varepsilon_0)\left(V_m^2 - m^2 u^2\right)}, \tag{28}$$

where we have used the notations

$$V_m = |\lambda_1|\rho_1 \frac{J_m'}{J_m} + \lambda_0 \rho_1 \frac{K_m'}{K_m}, \qquad u = \lambda_0/|\lambda_1| + |\lambda_1|/\lambda_0 \tag{29}$$

$$U_m = V_m \left(\varepsilon_0 |\lambda_1|\rho_1 \frac{J_m'}{J_m} + \varepsilon_1 \lambda_0 \rho_1 \frac{K_m'}{K_m}\right) - m^2 \frac{\lambda_0^2 + |\lambda_1|^2}{\lambda_0^2 |\lambda_1|^2}\left(\varepsilon_1 \lambda_0^2 + \varepsilon_0 |\lambda_1|^2\right). \tag{30}$$

Here and below it is understood $K_m = K_m(|\lambda_1|\rho_1)$, $J_m = J_m(\lambda_0 \rho_1)$ if the argument of the function is omitted, and the prime means the differentiation with respect to the argument of the function. Now the equation for the eigenmodes is written in the standard form (see, for instance, [16])

$$U_m = 0. \tag{31}$$

Unlike to the case of the waveguide with perfectly conducting walls, here the eigenmodes with $m \neq 0$ are not decomposed into independent TE and TM parts. This decomposition takes place only for the $m = 0$ mode and this case will be discussed below separately.

We denote by $\lambda_0 \rho_1 = \lambda_{m,s}$, $s = 1, 2, \ldots$, the solutions to equation (31) with respect to $\lambda_0 \rho_1$ for the modes with $m \neq 0$. The corresponding modes $k_z = k_{m,s}^{(\pm)}$ are related to these solutions by the formula

$$k_{m,s}^{(\pm)} = \frac{m\omega_0 \sqrt{\varepsilon_0}}{c\left(1 - \beta_{0\|}^2\right)}\left[\beta_{0\|} \pm \sqrt{1 + b_{m,s}^2\left(\beta_{0\|}^2 - 1\right)}\right], \qquad b_{m,s} = \frac{c\lambda_{m,s}}{m\omega_0 \rho_1 \sqrt{\varepsilon_0}}. \tag{32}$$

These modes are real under the condition $b_{m,s}^2\left(1-\beta_{0\|}^2\right) \leqslant 1$. For $\beta_{0\|} < 1$, the condition for $k_{m,s}^{(\pm)}$ to be real defines the maximum value for s, which we will denote by s_m:

$$\lambda_{m,s_m} < \frac{m\omega_0\rho_1\sqrt{\varepsilon_0}}{c\sqrt{1-\beta_{0\|}^2}} < \lambda_{m,s_m+1}. \tag{33}$$

If the Cherenkov condition $\beta_{0\|} > 1$ is satisfied, the upper limit s_m for s is determined by the dispersion law for the dielectric permittivity via the condition $\varepsilon_0(\omega_m) > c^2/v_\|^2$. The function $\omega_m(k_z)$ is also expressed in terms of $\lambda_{m,s}$:

$$\omega_m(k_z) = \frac{m\omega_0}{1-\beta_{0\|}^2}\left[1 \pm \beta_{0\|}\sqrt{1+b_{m,s}^2\left(\beta_{0\|}^2-1\right)}\right]. \tag{34}$$

In order to obtain unambiguous result for the fields we should specify the integration contour in the complex plane k_z in equation (12). For this we note that in physically real situations the dielectric permittivity ε_0 is a complex quantity: $\varepsilon_0 = \varepsilon_0' + \mathrm{i}\varepsilon_0''$. Assuming that ε_0'' is small, this induces an imaginary part for k_z given by formula

$$\mathrm{Im}\, k_z = \pm\frac{\left[1 \pm \beta_{0\|}\sqrt{1+b_{m,s}^2\left(\beta_{0\|}^2-1\right)}\right]^2}{2\sqrt{\varepsilon_0}\left(1-\beta_{0\|}^2\right)^2\sqrt{1+b_{m,s}^2\left(\beta_{0\|}^2-1\right)}}\mathrm{i}\varepsilon_0''(\omega_m), \tag{35}$$

where the coefficient and the argument of $\varepsilon_0''(\omega_m)$ are evaluated with the real part of dielectric permittivity. Note that one has $\varepsilon_0''(\omega_m) \gtrless 0$ for $\omega_m \gtrless 0$. In the case $\beta_{0\|} < 1$ from (34) we have $\omega_m > 0$ and, hence, in accordance with (35), $\mathrm{Im}\, k_z \gtrless 0$ for the modes $k_{m,s}^{(\pm)}$. Deforming the integration contour we see that for $\beta_{0\|} < 1$ in the integral over k_z in (13), the contour avoids the poles $k_{m,s}^{(-)}$ from above and the poles $k_{m,s}^{(+)}$ from below. For the case $\beta_{0\|} > 1$ we have $\omega_m < 0$ for the modes with $k_z = k_{m,s}^{(+)}$ and $\omega_m > 0$ for the modes with $k_z = k_{m,s}^{(-)}$ and, hence, in accordance with formula (35) for both types of modes $\mathrm{Im}\, k_z < 0$. As a result, in this case in the integral over k_z all poles $k_{m,s}^{(\pm)}$ should be avoided from above.

Now let us consider the electromagnetic fields inside the cylinder for large distances from the charge. In this case the main contribution into the fields comes from the poles of the integrand. Having specified the integration contour now we can evaluate the radiation parts of the fields inside the cylinder. Closing the integration contour by the large semicircle, under the condition $\beta_{0\|} < 1$ one finds

$$F_l^{(\mathrm{rad})}(\mathbf{r},t) = \sigma F_l^{(\sigma)}(\mathbf{r},t) = \sigma 4\pi\, \mathrm{Re}\left[\mathrm{i}\sum_{m=1}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\omega_0 t)}\sum_{s=1}^{s_m}\underset{k_z=k_{m,s}^{(\sigma)}}{\mathrm{Res}}\,\mathrm{e}^{\mathrm{i}k_z(z-v_\| t)}F_{ml}(k_z,\rho)\right], \tag{36}$$

where $\sigma = +(-)$ for $z - v_\| t > 0$ $(z - v_\| t < 0)$. This expression describes waves propagating along the positive direction of the axis z for $\sigma = +$ and for $\sigma = -$, $b_{m,s} < 1$, and waves propagating along the negative direction to the axis z for $\sigma = -$, $1 < b_{m,s} < 1/\sqrt{1-\beta_{0\|}^2}$. Under the condition $\beta_{0\|} > 1$ for the radiation fields one finds

$$F_l^{(\mathrm{rad})}(\mathbf{r},t) = -\sum_{\sigma=\pm}F_l^{(\sigma)}(\mathbf{r},t)\theta(v_\| t - z), \tag{37}$$

where $\theta(x)$ is the Heaviside unit step function. As we could expect, in this case the radiation field is behind the charge. Formula (37) describes the waves propagating along the positive direction of the axis z for $\sigma = +$ and for $\sigma = -$, $b_{m,s} > 1$, and waves propagating along the negative direction of the axis z for $\sigma = -$, $b_{m,s} < 1$. Note that for $b_{m,s} > 1$ there are no waves propagating along the negative direction of the axis z.

By evaluating the residue in (36), for the functions $F_l^{(\sigma)}(\mathbf{r},t)$ one finds

$$F_l^{(\sigma)}(\mathbf{r},t) = 4\pi\,\mathrm{Re}\left[\mathrm{i}\sum_{m=1}^{\infty}\mathrm{e}^{\mathrm{i}m(\phi-\omega_0 t)}\sum_{s=1}^{s_m}\mathrm{e}^{\mathrm{i}k_z(z-v_\parallel t)}\frac{F_{ml}^{(\sigma)}(k_z,\rho)}{\mathrm{d}\alpha_m/\mathrm{d}k_z}\right]_{k_z=k_{m,s}^{(\sigma)}}, \tag{38}$$

where the formulae for the coefficients $F_{ml}^{(\sigma)}(k_z,\rho), l = \rho,\phi,z$ are obtained from the corresponding expressions for $F_{ml}^{(\sigma)}$ by the replacement $C_m^{(p)} \to C_m^{(p\sigma)}$ for the components of the vector potential and by the replacement $D_m^{(p)} \to C_m^{(p\sigma)}$ for the electric and magnetic fields, with

$$C_m^{(p\sigma)} = \frac{K_{m+p}K_m}{4pcW_{m+p}^{(\sigma)}}\left[4v_\parallel k_{m,s}^{(\sigma)}\rho_1\frac{J_m(\lambda_{m,s}\rho_{01})}{W_m^{(\sigma)}} - v_\perp\lambda_{m,s}^{(\sigma)}\sum_{l=\pm1}\frac{J_{m+l}(\lambda_{m,s}\rho_{01})}{lW_{m+l}^{(\sigma)}}\right], \tag{39}$$

where we have introduced the notation

$$W_m^{(\sigma)} = \lambda_{m,s}^{(\sigma)}J_m(\lambda_{m,s})K_m'\big(\lambda_{m,s}^{(\sigma)}\big) - \lambda_{m,s}J_m'(\lambda_{m,s})K_m\big(\lambda_{m,s}^{(\sigma)}\big), \tag{40}$$

and

$$\lambda_{m,s}^{(\pm)} = \sqrt{\left(1-\frac{\varepsilon_1}{\varepsilon_0}\right)k_{m,s}^{(\pm)2}\rho_1^2 - \frac{\varepsilon_1}{\varepsilon_0}\lambda_{m,s}^2}, \qquad \rho_{01} = \frac{\rho_0}{\rho_1}. \tag{41}$$

Note that we have the following relations:

$$W_{m+p}^{(\sigma)} = (pV_m - mu)J_mK_m, \qquad W_m^{(\sigma)} = \frac{k_{m,s}^{(\sigma)2}\rho_1^2}{\lambda_{m,s}\lambda_{m,s}^{(\sigma)}V_m}\big(V_m^2 - m^2u^2\big)J_mK_m. \tag{42}$$

Unfortunately, for the case $m \neq 0$ we have no summation formula over the eigenmodes like to that for the $m = 0$ mode (see below) and for the further evaluation of the radiation fields numerical methods have to be used.

Now we turn to the modes with $m = 0$. In this case there are two types of eigenmodes. The first ones are the zeros of the function α_0 corresponding to the zeros of the numerator in (25). The equation for these eigenmodes is written in the form

$$\varepsilon_0|\lambda_1|\frac{J_0'}{J_0} + \varepsilon_1\lambda_0\frac{K_0'}{K_0} = 0, \qquad \lambda_i = \lambda_i^{(0)}, \tag{43}$$

where $\lambda_i^{(0)}$ is defined by formula (24). For the presence of these modes we have the conditions $\beta_{1\parallel} < 1 < \beta_{0\parallel}$. In particular we should have $\varepsilon_1 < \varepsilon_0$. From formula (26) it follows that the combination $\sum_p pD_m^{(p)}$ has no poles at these eigenmodes and for the corresponding radiation field one has $H_{0\rho}^{(1)} = H_{0z}^{(1)} = 0$ and $E_{0\phi}^{(1)} = 0$. As a result, these waves are TM waves or the waves of the E-type. The values for k_z defined by equation (43) we will denote by $\pm k_{0,s}^{(\mathrm{E})}$, $s = 1, 2, \ldots$. The second type of $m = 0$ eigenmodes corresponds to the zeros of the function W_1^J. The corresponding equation can be written in the form

$$\lambda_0\frac{K_0'}{K_0} + |\lambda_1|\frac{J_0'}{J_0} = 0, \qquad \lambda_i = \lambda_i^{(0)}. \tag{44}$$

These zeros are poles for the combination $\sum_p pD_m^{(p)}$, whereas the combination $\sum_p D_m^{(p)}$ is regular at these zeros. For the corresponding radiation fields one has $H_{0\phi}^{(1)} = 0$ and $E_{0\rho}^{(1)} = E_{0z}^{(1)} = 0$ and these waves are TE waves or the waves of the M-type. The corresponding eigenvalues for k_z we will denote by $\pm k_{0,s}^{(\mathrm{M})}$, $s = 1, 2, \ldots$. Note that both types of the eigenmodes are also obtained from (31) taking $m = 0$.

As in the case of $m \neq 0$, to define the fields from formula (13) we should specify the contour for the integration in the complex plane k_z. In the way similar to that used for the modes with $m \neq 0$, it can be seen that the contour avoids the poles $\pm k_{0,s}^{(\mathrm{F})}$ from above. The corresponding radiation fields are found by evaluating the k_z-integrals with the help of residue theorem. For $z - v_\parallel t > 0$ we close the integration contour in the upper half-plane of the complex variable k_z and the radiation fields are zero. We could expect this result as $\beta_{0\parallel} > 1$ and the radiation field is behind the charge. For the points in the region $z - v_\parallel t < 0$ we close the integration contour in the lower half-plane and the integral is equal to the residues at the poles $\pm k_{0,s}^{(\mathrm{E})}$ and $\pm k_{0,s}^{(\mathrm{M})}$ multiplied by $-2\pi\mathrm{i}$. Here we will present the formulae for the nonzero components of the electric field:

$$
\begin{aligned}
E_{\rho,m=0}^{(\mathrm{E})} &= \frac{4q}{\rho_1^2}\sum_s \frac{\varepsilon_1\sqrt{\beta_{0\parallel}^2-1}}{\varepsilon_1-\varepsilon_0}\frac{J_0\big(\rho_0\lambda_{0,s}^{(\mathrm{E})}\big)J_1\big(\rho\lambda_{0,s}^{(\mathrm{E})}\big)\sin\big[k_{0,s}^{(\mathrm{E})}(z-v_\parallel t)\big]}{\varepsilon_0 J_0'^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)+\varepsilon_1\big(\beta_{0\parallel}^2-1\big)J_0^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)},\\
E_{z,m=0}^{(\mathrm{E})} &= \frac{4q}{\rho_1^2}\sum_s \frac{\varepsilon_1\big(\beta_{0\parallel}^2-1\big)}{\varepsilon_1-\varepsilon_0}\frac{J_0\big(\rho_0\lambda_{0,s}^{(\mathrm{E})}\big)J_0\big(\rho\lambda_{0,s}^{(\mathrm{E})}\big)\cos\big[k_{0,s}^{(\mathrm{E})}(z-v_\parallel t)\big]}{\varepsilon_0 J_0'^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)+\varepsilon_1\big(\beta_{0\parallel}^2-1\big)J_0^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)},\\
E_{\phi,m=0}^{(\mathrm{M})} &= \frac{4qv_\perp}{\rho_1^2 v_\parallel}\sum_s \frac{J_1\big(\rho_0\lambda_{0,s}^{(\mathrm{M})}\big)J_1\big(\rho\lambda_{0,s}^{(\mathrm{M})}\big)}{(\varepsilon_1-\varepsilon_0)J_0'^2\big(\rho_1\lambda_{0,s}^{(\mathrm{M})}\big)}\cos\big[k_{0,s}^{(\mathrm{M})}(z-v_\parallel t)\big],
\end{aligned}
\tag{45}
$$

where we have introduced the notation

$$
\lambda_{0,s}^{(\mathrm{F})} = k_{0,s}^{(\mathrm{F})}\sqrt{\beta_{0\parallel}^2-1}, \qquad \mathrm{F}=\mathrm{E},\mathrm{M}. \tag{46}
$$

In these formulae the summands with a given s describe the radiation field with the frequency $\omega_{0,s}^{(\mathrm{F})} = v_\parallel k_{0,s}^{(\mathrm{F})}$. In the appendix it is shown that $\lambda_{0,s}^{(\mathrm{F})}\rho_1 \gtrsim 1$ and, hence, for the corresponding frequencies one has the estimate $\omega_{0,s}^{(\mathrm{F})} \gtrsim v_\parallel\big/\big(\rho_1\sqrt{\beta_{0\parallel}^2-1}\big)$. In formulae (45) the upper limit for the summation over s, which we will denote by s_m, is determined by the dispersion law of the dielectric permittivity $\varepsilon = \varepsilon(\omega)$ through the condition $\varepsilon\big(\omega_{0,s}^{(\mathrm{F})}\big) > c^2\big/v_\parallel^2$.

Having the electric field we can evaluate the radiation intensity on the mode $m = 0$ inside the dielectric waveguide by using the formula

$$
I_{m=0} = -\int (j_\phi E_{\phi,m=0} + j_z E_{z,m=0})\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z. \tag{47}
$$

For this quantity one finds

$$
I_{m=0} = I_{m=0}^{(\mathrm{E})} + I_{m=0}^{(\mathrm{M})}, \tag{48}
$$

where for the contributions of the TM and TM waves we have

$$
I_{m=0}^{(\mathrm{E})} = \frac{2q^2 v_\parallel}{\rho_1^2}\sum_s \frac{\beta_{0\parallel}^2-1}{\varepsilon_0-\varepsilon_1}\frac{J_0^2\big(\rho_0\lambda_{0,s}^{(\mathrm{E})}\big)}{(\varepsilon_0/\varepsilon_1)J_0'^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)+\big(\beta_{0\parallel}^2-1\big)J_0^2\big(\rho_1\lambda_{0,s}^{(\mathrm{E})}\big)}, \tag{49a}
$$

$$
I_{m=0}^{(\mathrm{M})} = \frac{2q^2 v_\perp^2}{\rho_1^2 v_\parallel}\sum_s \frac{J_1^2\big(\rho_0\lambda_{0,s}^{(\mathrm{M})}\big)}{\big(\varepsilon_0-\varepsilon_1\big)J_0'^2\big(\rho_1\lambda_{0,s}^{(\mathrm{M})}\big)}. \tag{49b}
$$

As the modes $k_{0,s}^{(\mathrm{F})}$ are not explicitly known as functions on s, formulae (49*a*), (49*b*) are not convenient for the evaluation of the corresponding radiation intensities. More convenient form may be obtained by making use of the summation formula (A.8) derived in the appendix, taking

$$
\eta = \sqrt{\frac{1-\beta_{1\parallel}^2}{\beta_{0\parallel}^2-1}}. \tag{50}
$$

In formula (A.8) we choose $\alpha = \varepsilon_0/\varepsilon_1$, $f(z) = zJ_0^2(z\rho_0/\rho_1)$ for the waves of the E-type and $\alpha = 1$, $f(z) = zJ_1^2(z\rho_0/\rho_1)$ for the waves of the M-type. For both types of modes one has $f(-\mathrm{i}x) = -f(\mathrm{i}x)$ and we can use the version of the summation formula given by (A.10). In the intermediate step of the calculations, it is technically simpler instead of considering the dispersion of the dielectric permittivity to assume that in formulae (49) a cutoff function $\psi_\mu\big(\lambda_{0,s}^{(\mathrm{F})}\big)$ is introduced with μ being the cutoff parameter and $\psi_0 = 1$ [for example, $\psi_\mu(x) = \exp(-\mu x)$], which will be removed after the summation. In this way, after the application of the summation formula we find the following results:

$$I_{m=0}^{(\mathrm{E})} = q^2 v_\parallel \left[c^2 \int_0^\infty \mathrm{d}x \frac{x}{\varepsilon_0} \psi_\mu(x) J_0^2(\rho_0 x) + \frac{4}{\pi^2\rho_1^2} \int_0^\infty \mathrm{d}x \frac{I_0^2(x\rho_0/\rho_1)}{\varepsilon_1 x g_{\varepsilon_0/\varepsilon_1}(\eta, x)}\right], \tag{51a}$$

$$I_{m=0}^{(\mathrm{M})} = \frac{q^2 v_\perp^2 v_\parallel}{c^2} \left[\int_0^\infty \mathrm{d}x \frac{x\psi_\mu(x)}{\beta_{0\parallel}^2 - 1} J_1^2(\rho_0 x) - \frac{4}{\pi^2\rho_1^2} \int_0^\infty \mathrm{d}x \frac{I_1^2(x\rho_0/\rho_1)}{\big(\beta_{0\parallel}^2 - 1\big) x g_1(\eta, x)}\right], \tag{51b}$$

where the function $g_\alpha(\eta, x)$ is defined by formula (A.11). In the first terms in the square brackets replacing the integration variable by the frequency $\omega = v_\parallel x/\sqrt{\beta_{0\parallel}^2 - 1}$ and introducing the physical cutoff through the condition $\beta_{0\parallel} > 1$ instead of the cutoff function, we see that these terms coincide with the radiation intensities on the harmonic $m = 0$ for the waves of the E- and M-type in the homogeneous medium with dielectric permittivity ε_0 (see [14]). The second terms in the square brackets are induced by the inhomogeneity of the medium in the problem under consideration. Note that, unlike to the terms corresponding to the homogeneous medium, for $\rho_0 < \rho_1$ the latter are finite also in the case when the dispersion is absent: for large values of x the integrands decay as $\exp[-2x(1 - \rho_0/\rho_1)]$ (for this reason we have removed the cutoff function from these terms). In particular, from the last observation it follows that under the condition $(1-\rho_0/\rho_1) \gg v_\parallel/\omega_d$, where ω_d is the characteristic frequency for the dispersion of the dielectric permittivity, the influence of the dispersion on the inhomogeneity induced terms can be neglected. Note that in a homogeneous medium the corresponding radiation propagates under the Cherenkov angle $\theta_C = \arccos(1/\beta_{0\parallel})$ and has a continuous spectrum, whereas the radiation described by (51) propagates inside the dielectric cylinder and has a discrete spectrum with frequencies $\omega_{0,s}^{(\mathrm{F})}$. As the function $g_\alpha(\eta, x)$ is always nonnegative, from formulae (51) we conclude that the presence of the cylinder amplifies the $m = 0$ part of the radiation for the waves of the E-type and suppresses the radiation for the waves of the M-type. In the helical undulators one has $v_\perp \ll v_\parallel$ and the contribution of the TM waves dominates.

In addition to the total intensity, it is of interest to have the distribution of the intensity over the frequencies. To this aim we define the partial intensities $I_{m=0,s}^{(\mathrm{F})}$ for a given s related to the quantity $I_{m=0}^{(\mathrm{F})}$ by the formula $I_{m=0}^{(\mathrm{F})} = \sum_s I_{m=0,s}^{(\mathrm{F})}$. In figure 1 we have presented the results of the numerical evaluation for the number of the quanta radiated per unit time, $N_{m=0,s}^{(\mathrm{F})} = I_{m=0,s}^{(\mathrm{E})}\big/\hbar\omega_{0,s}^{(\mathrm{F})}$, on a harmonic with a given s, as a function of s for the electron with the energy 2 MeV moving inside a cylinder with dielectric permittivity $\varepsilon_0 = 3$. It is assumed that for the exterior region $\varepsilon_1 = 1$. The angle θ_0 between the electron velocity and the cylinder axis ($\tan\theta_0 = v_\perp/v_\parallel$) is taken 0.1 rad, and $\rho_0/\rho_1 = 0.5$. The values of the parameter $\rho_1\lambda_{0,s}^{(\mathrm{F})}$ for the first eight harmonics are presented in the table. The corresponding frequencies are related to this parameter by the formula $\omega_{0,s}^{(\mathrm{F})} = v_\parallel\lambda_{0,s}^{(\mathrm{E})}\big/\sqrt{\beta_{0\parallel}^2 - 1}$.

s	1	2	3	4	5	6	7	8
$\rho_1\lambda_{0,s}^{(\mathrm{E})}$	2.728	5.931	9.102	12.262	15.415	18.565	21.713	24.860
$\rho_1\lambda_{0,s}^{(\mathrm{M})}$	2.507	5.653	8.801	11.947	15.092	18.236	21.380	24.523

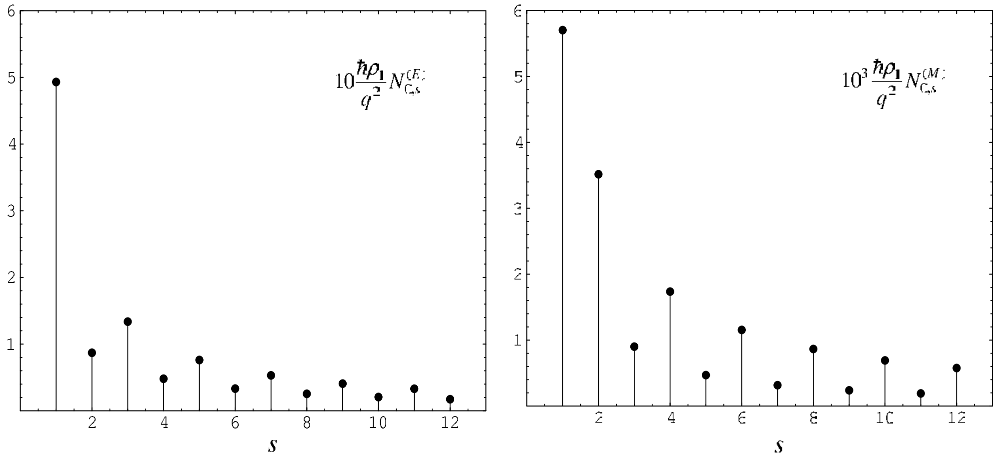

Figure 1. The number of the radiated quanta on the mode $m = 0$ as a function of the harmonic number s for the waves of the E- (left panel) and M-type (right panel) (for the values of the parameters see the text).

4. Conclusion

We have considered the electromagnetic field generated by a charged particle moving along a helical orbit inside a dielectric cylinder immersed into a homogeneous medium. The fields and the spectral-angular distribution of the radiation intensity in the exterior medium have been investigated in our previous paper [14]. In particular, the conditions were specified under which strong narrow peaks appear in the angular distribution for the number of radiated quanta. In the present paper we study the fields inside the cylinder. By using the corresponding formulae for the components of the Green function, we have derived expressions for the electromagnetic potentials and fields. These expressions are presented in the form of sums of two terms. The first ones correspond to the fields generated by the charge in a homogeneous medium with the same dielectric permittivity as that for the material of the cylinder. The second terms are induced by the difference of the dielectric permittivities for the exterior and interior media, and are given by formulae (16), (21), (23). We have extracted the parts of the fields which are responsible for the radiation. For the radiation propagating inside the cylinder the projection of the wave vector on the cylinder axis takes discrete set of values which are solutions to the eigenmode equation (31). The expressions for the electromagnetic fields corresponding to the radiation propagating inside the cylinder are derived. As an application we have considered the radiation intensity on the mode $m = 0$. In this case the radiation field is separated into purely TE and TM modes and we have found the intensities for both types of these modes determined by formulae (49). These formulae presents the intensities as sums over the modes of the dielectric waveguide. However, as the corresponding eigenmodes are not explicitly known, this form is not convenient for the numerical evaluation of the radiation intensity. In the appendix, by using the generalized Abel–Plana formula, we derive a summation formula for the series over the normal modes of the dielectric cylinder which allows us to extract from the radiation intensity the parts corresponding to the radiation in a homogeneous medium and to present the inhomogeneity-induced part in terms of exponentially converging integrals. Unlike to the parts corresponding to the homogeneous medium, the terms induced by the presence of the cylinder are finite also in the case when the dispersion is absent. We have specified the condition under which the influence of dispersion on the inhomogeneity-induced effects can be neglected.

The type of motion discussed in the present paper is involved in magnetic devices called helical undulators which are inserted into a straight sector of storage rings. The helical undulators are used to generate circularly polarized intense electromagnetic radiation in a relatively narrow bandwidth. The frequency of radiation is tunable by varying the beam energy and the magnetic field. As we have seen in the present paper, the insertion of a dielectric waveguide into the helical undulator provides an additional mechanism for tuning the characteristics of the emitted radiation by choosing the parameters of the waveguide. The radiated energy inside the cylinder is redistributed among the cylinder modes and, as a result, the corresponding spectrum differs significantly from the homogeneous medium or free-space results. This change is of special interest in the low-frequency range where the distribution of the radiation energy among small number of modes leads to the enhancement of the spectral density for the radiation intensity. The radiation emitted on the waveguide modes propagates inside the cylinder and the waveguide serves as a natural collector for the radiation. This eliminates the necessity for focusing to achieve a high-power spectral intensity. The geometry considered here is of interest also from the point of view of generation and transmitting of waves in waveguides, a subject which is of considerable practical importance in microwave engineering and optical fibre communications.

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Appendix. Summation formula over the eigenmodes

In this section we derive a summation formula for the series over zeros of the function

$$C_\alpha(\eta, z) = V_\alpha\{J_0(z), K_0(\eta z)\}, \tag{A.1}$$

where and in what follows for given functions $F(z)$ and $G(z)$ we use the notation

$$V_\alpha\{F(z), G(\eta z)\} = F(z)G'(\eta z) + \alpha\eta F'(z)G(\eta z), \tag{A.2}$$

with $\alpha \geqslant 1$ and η being real constants. For this we will use the generalized Abel–Plana formula from [17] (for applications of the generalized Abel–Plana formula in quantum field theory with boundaries see [18]). In this formula we substitute

$$g(z) = \mathrm{i} f(z)\frac{V_\alpha\{Y_0(z), K_0(\eta z)\}}{C_\alpha(\eta, z)}. \tag{A.3}$$

For the combinations of the functions entering in the generalized Abel–Plana formula one has

$$f(z) - (-1)^k g(z) = f(z)\frac{V_\alpha\left\{H_0^{(k)}(z), K_0(\eta z)\right\}}{C_\alpha(\eta, z)}, \tag{A.4}$$

where $H_\nu^{(k)}(z)$, $k = 1, 2$, are the Hankel functions. Let us denote positive zeros of the function $C_\alpha(\eta, z)$ by k_s, $s = 1, 2, \ldots,$ assuming that these zeros are arranged in the ascending order. Note that, for $z \gg 1$ we have $C_\alpha(\eta, z) \approx K_0'(\eta z) - \alpha\eta z K_0(\eta z)/2 < 0$ and, hence, $k_s \gtrsim 1$. By using the asymptotic formulae for the cylindrical functions for large values of the argument, it can be seen that for large values s one has $k_s \approx -\arctan(1/\alpha\eta) + \pi/4 + \pi s$. For the derivative of the function $C_\alpha(\eta, z)$ at the zeros k_s one obtains

$$C'_\alpha(\eta, z) = -\eta \frac{K_0(\eta z)}{J_0(z)} \left[\alpha(1 + \alpha\eta^2) J_0'^2(z) + (\alpha - 1) J_0^2(z)\right], \qquad z = k_s. \tag{A.5}$$

In particular, it follows from here that the zeros are simple. Assuming that the function $f(z)$ is analytic in the right half-plane, for the residue term in the generalized Abel–Plana formula one finds

$$\operatorname*{Res}_{z=k_s} g(z) = -\frac{\mathrm{i}}{\pi} T_\alpha(k_s) f(k_s), \tag{A.6}$$

where we have introduced the notation

$$T_\alpha(z) = \frac{2\alpha/z}{\alpha(1 + \alpha\eta^2) J_0'^2(z) + (\alpha - 1) J_0^2(z)}. \tag{A.7}$$

Substituting the expressions for the separate terms into the generalized Abel–Plana formula we obtain the following result:

$$\begin{aligned} \lim_{x_0\to\infty} \left[\sum_{s=1}^{s_0} T_\alpha(k_s) f(k_s) - \int_0^{x_0} \mathrm{d}x\, f(x)\right] &= -\frac{1}{\pi} \int_0^\infty \mathrm{d}x \left[f(\mathrm{i}x) \frac{V_\alpha\{K_0(x), H_0^{(2)}(\eta x)\}}{V_\alpha\{I_0(x), H_0^{(2)}(\eta x)\}} \right. \\ &\quad \left. + f(-\mathrm{i}x) \frac{V_\alpha\{K_0(x), H_0^{(1)}(\eta x)\}}{V_\alpha\{I_0(x), H_0^{(1)}(\eta x)\}} \right], \end{aligned} \tag{A.8}$$

where s_0 is defined by the relation $k_{s_0} < x_0 < k_{s_0+1}$. This formula is valid for functions $f(z)$ obeying the condition

$$|f(z)| < \epsilon(x)\, \mathrm{e}^{c|y|}, \qquad z = x + \mathrm{i}y, \qquad |z| \to \infty, \tag{A.9}$$

where $c < 2$ and $\epsilon(x) \to 0$ for $x \to \infty$. Formula (A.8) is further simplified for functions satisfying the additional condition $f(-\mathrm{i}x) = -f(\mathrm{i}x)$:

$$\lim_{x_0\to\infty} \left[\sum_{s=1}^{s_0} T_\alpha(k_s) f(k_s) - \int_0^{x_0} \mathrm{d}x\, f(x)\right] = -\frac{4\mathrm{i}\alpha}{\pi^2} \int_0^\infty \mathrm{d}x \frac{f(\mathrm{i}x)}{x^2 g_\alpha(\eta, x)}, \tag{A.10}$$

where

$$\begin{aligned} g_\alpha(\eta, x) &= I_0^2(x)\left[J_1^2(\eta x) + Y_1^2(\eta x)\right] + \alpha^2\eta^2 I_1^2(x)\left[J_0^2(\eta x) + Y_0^2(\eta x)\right] \\ &\quad - 2\alpha\eta I_0(x) I_1(x)[J_0(\eta x) J_1(\eta x) + Y_0(\eta x) Y_1(\eta x)]. \end{aligned} \tag{A.11}$$

Note that we have denoted $g_\alpha(\eta, x) = |V\{I_0(z), H_0^{(1)}(\eta z)\}|^2$ and this function is always nonnegative.

References

[1] 1999 *Synchrotron Radiation Theory and Its Development* ed V A Bordovitsyn (Singapore: World Scientific)

[2] Hofman A 2004 *The Physics of Sinchrotron Radiation* (Cambridge: Cambridge University Press)

[3] Grigoryan L Sh, Kotanjyan A S and Saharian A A 1995 *Izv. Akad. Nauk Arm. SSR Fiz.* **30** 239
Grigoryan L Sh, Kotanjyan A S and Saharian A A 1995 *Sov. J. Contemp. Phys.* **30** 1 (Engl. Transl.)

[4] Arzumanian S R, Grigoryan L Sh, Kotanjyan Kh V and Saharian A A 1995 *Izv. Akad. Nauk Arm. SSR Fiz.* **30** 106
Arzumanian S R, Grigoryan L Sh, Kotanjyan Kh V and Saharian A A 1995 *Sov. J. Contemp. Phys.* **30** 12 (Engl. Transl.)

[5] Grigoryan L Sh, Khachatryan H F and Arzumanyan S R 1998 *Izv. Akad. Nauk Arm. SSR Fiz.* **33** 267
Grigoryan L Sh, Khachatryan H F and Arzumanyan S R 1998 *Sov. J. Contemp. Phys.* **33** 1 (*Preprint* cond-mat/0001322) (Engl. Transl.)

[6] Kotanjyan A S, Khachatryan H F, Petrosyan A V and Saharian A A 2000 *Izv. Akad. Nauk Arm. SSR Fiz.* **35** 115
Kotanjyan A S, Khachatryan H F, Petrosyan A V and Saharian A A 2000 *Sov. J. Contemp. Phys.* **35** 1 (Engl. Transl.)

[7] Kotanjyan A S and Saharian A A 2001 *Izv. Akad. Nauk Arm. SSR Fiz.* **36** 310
Kotanjyan A S and Saharian A A 2001 *Sov. J. Contemp. Phys.* **36** 7 (Engl. Transl.)

[8] Grigoryan L Sh, Khachatryan H F and Arzumanyan S R 2002 *Izv. Akad. Nauk Arm. SSR Fiz.* **37** 327
Grigoryan L Sh, Khachatryan H F and Arzumanyan S R 2002 *Sov. J. Contemp. Phys.* **37** 1 (Engl. Transl.)

[9] Kotanjyan A S and Saharian A A 2002 *Mod. Phys. Lett.* A **17** 1323

[10] Kotanjyan A S 2003 *Nucl. Instrum. Methods* B **201** 3

[11] Grigoryan L Sh, Khachatryan H F, Arzumanyan S R and Grigoryan M L 2006 *Nucl. Instrum. Methods* B **252** 50

[12] Saharian A A and Kotanjyan A S 2003 *Izv. Akad. Nauk Arm. SSR Fiz.* **38** 288
Saharian A A and Kotanjyan A S 2003 *Sov. J. Contemp. Phys.* **38** 5 (Engl. Transl.)

[13] Saharian A A and Kotanjyan A S 2004 *Nucl. Instrum. Methods.* B **226** 351

[14] Saharian A A and Kotanjyan A S 2005 *J. Phys. A: Math. Gen.* **38** 4275

[15] Gevorgian L A and Poghosian P M 1984 *Izv. Akad. Nauk Arm. SSR Fiz.* **19** 239

[16] Jackson J D 1998 *Classical Electrodynamics* (New York: Wiley)

[17] Saharian A A 2000 The generalized Abel-Plana formula. Applicatons to Bessel functions and Casimir effect *Report No* IC/2000/14 *Preprint* hep-th/0002239

[18] Saharian A A 2006 Generalized Abel-Plana formula as a renormalization tool in quantum field theory with boundaries *PoS IC2006*, 019 (*Preprint* hep-th/0609093)